IJSRIS journal
INTERNATIONAL JOURNAL OF SCIENTIFIC RESEARCH AND INNOVATIVE STUDIES
ISSN: 2820-7157
www.ijsrisjournal.com

October 2023 Volume 2 Number 3 Pages 12-16

HILBERT SPACE AND UNIQUE COMMON FIXED POINT FOR SELF MAPPINGS

Shoyeb Ali Sayyed*, Raj Singh Chandelkar And S.K.Jain*****

* Professor, Department of Mathematics, Malwanchal University, Indore(M.P.)India
shoyeb9291@gmail.com
** Assistant Professor, Department of Mathematics , Govt. Model College , Jhabua (M.P.)-457661 India
rajchandelkar2910@gmail.com
*** Professor, Department Of Applied Mathematics, Ujjain Engineering College Ujjain (M.P.) India
Skjain63engg@Gmail.Com

ABSTRACT

In present paper , enhanced the outcome of prominent educators as well as analyzers. Using self mappings along with functional type inequality in Hilbert space for building up desired unique common fixed point. For procuring we have done some tempering in old survey. Our desired result of the theorem is affecting by a great number of experts.

KEY WORDS: Closed Convex Subset , Functional Inequality, Hilbert Space, Ishikawa Iteration, Self Mappings, Unique Common Fixed Point

1. INTRODUCTION AND PRELIMINARY

Imdad and Jawed [3]observed that the general form of his theorem remains true in metric spaces. In doing this, never needed the specific properties of an inner product norm. A real or complex inner product space (RIPS or CIPS) i.e. also a complete metric space (CMS) with respect to distance function pressed by inner product is a Hilbert space. Sayyed and Badshah [14,15,16,17] gave more significant path to newcomers with road map of future enhancements . In same pattern Jain and Sayyed [4] gave result on nonlinear contraction with gaining a fixed point as well as Sayyed et.al.[18,19] examined it for self maps. For consequently started with Ciric [1], Das and Gupta [2], Naimpally and Singh [6], Rhoades [12,13], Yadav et,al [21], Veerapandi and Kumar [20],Rao et.al.[10] , Rao and Kalyani [11], Patel and Sharma [9].Nigam et.al. [7] and Park [8].

For elaborating proper some of modifications have been created which are as follows :

(i) In the Ishikawa scheme $\{\mu_{2n}\}$, $\{\omega_{2n}\}$ satisfies $0 \le \mu_{2n}$, $\omega_{2n} \le 1$, $0 \le \mu_{2n}$, $\omega_{2n} \le 1$ for all n,

$$\lim \omega_{2n} = 0 \text{ as } n \to \infty \text{ and } \sum \mu_{2n}\, \omega_{2n} = \infty.$$

(ii) $\lim_{n\to\infty} \mu_{2n} = \mu_0 > 0$

(iii)) $\lim_{n\to\infty} \omega = \omega_0 > 1$.

Let C be a non empts subset of B, where B is a Banach space and U and V: C → C be two mappings. The iteration scheme, called I- Scheme, defined as follows:

$r_0 \in C$

(1.1)

$s_{2n} = \omega_{2n}\ Ur_{2n} + (1-\omega_{2n})\ r_{2n} \qquad n \geq 0$

$r_{2n+1} = (1-\mu_{2n})\ r_{2n} + \mu_{2n}\ Vs_{2n} \qquad n \geq 0$

.... (1.2)

$s_{2n+1} = \omega_{2n+1}\ Ur_{2n+1} + (1-\omega_{2n+1})\ r_{2n+1} \qquad n \geq 0$

$r_{2n+1} = (1-\mu_{2n+1})\ r_{2n+1} + \mu_{2n+1}\ Vs_{2n+1} \qquad n \geq 0$

(1.3)

We know that Banach Space is Hilbert space if and onls if its norms satisfies the parallelogram law,

i.e. for every r, s $\in X$.

$$||r+s||^2 + ||r-s||^2 = 2\ ||r||^2 + 2||s||^2 \tag{1.4}$$

which implies, $||r+s||^2 \leq 2\ ||r||^2 + 2||s||^2$

(1.5)

2. MAIN RESULTS

THEOREM 2.1 : Let B be a Hilbert space and let C be a closed convex subset or B. Let U and V be two mappings satisfying with r,s are non negative and

$0 \leq 3a+3b+3c \leq 1$,

$$||Ur-Vs||^2 \leq a\ \frac{||s-Vs||^2[1+||r-Ur||^2]}{1+||r-s||^2} + b\ \frac{1+||s-Vs||^2||r-Ur||^2}{1+||r-s||^2} + c\ ||r-s||^2 \tag{2.1}$$

If there exists a point r_0 such that the I- scheme for U and V defined by (1.2) and (1.3), converges to a point u, then u is a common point of U and V.

PROOF:

It follows from (1.2) that $r_{2n+1} - r_{2n} = \mu_{2n}\ (Vs_{2n} - r_{2n})$

Since $r_{2n} \to u$, $||\ r_{2n+1} - r_{2n}\ || \to \infty$

Since $\{\ \mu_{2n}\ \}$ is bounded away from zero,

$||Vs_{2n} - r_{2n}|| \to 0$ as $n \to \infty$.

It follows that $||u - Vs_{2n}\ || \to 0$ as $n \to \infty$.

Since U and V satisfy (2.1) we have

$$||Ur_{2n}-Vs_{2n}||^2 \leq a\ \frac{||s-Vs_{2n}||^2[1+||r-Ur_{2n}||^2]}{1+||r-s||^2} + b\ \frac{[1+||s_{2n}-Vs_{2n}||^2\ ]||r_{2n}-Ur_{2n}||^2}{1+||r_{2n}-s_{2n}||^2} + c\ ||r_{2n}-s_{2n}||^2 \tag{2.2}$$

Now, $||s_{2n} - r_{2n}||^2 = ||\ \omega_{2n}\ Ur_{2n} + (1-\omega_{2n})\ r_{2n} - r_{2n}\ ||^2$

$= ||\ \omega_{2n}\ Ur_{2n} + r_{2n} - \omega_{2n} r_{2n} - c\ ||^2$

$= ||\ \omega_{2n}\ (Ur_{2n} - r_{2n})\ ||^2$

$= \omega^2_{2n}\ ||\ (Ur_{2n} + Vs_{2n}) + (Vs_{2n} - r_{2n})\ ||^2$

$\leq 2\ ||Ur_{2n} - Vs_{2n}||^2 + 2\ ||\ Vs_{2n} - r_{2n}||^2$

(2.3)

and

$||s_{2n} - Vs_{2n}||^2 = ||\ \omega_{2n}\ Ur_{2n} + (1-\omega_{2n})\ r_{2n} - Vs_{2n}||^2$

$= ||\ \omega_{2n}\ Ur_{2n} + (1-\omega_{2n})\ r_{2n} - Vs_{2n} + \omega_{2n}$

$Vs_{2n} - \omega s_{2n}\ Vs_{2n}\ ||^2$

$= ||\ \omega_{2n}\ (Ur_{2n} - Vs_{2n}) + (1-\omega_{2n})\ (r_{2n} - Vs_{2n})||^2$

$\leq 2\ \omega^2_{2n}\ ||\ Ur_{2n} - Vs_{2n}||^2 + 2\ (1-\omega_{2n})^2\ ||\ r_{2n} - Vs_{2n}||^2$

$\leq 2\ ||\ Ur_{2n} - Vs_{2n}||^2 + 2\ ||\ r_{2n} - Vs_{2n}||^2$.

(2.4)

from (2.2), (2.3) , (2.4) can be written as:

$$||Ur_{2n}-Vs_{2n}||^2 \leq a\ \frac{[2||Ur_{2n}-Vs_{2n}||^2] + 2\ ||r_{2n}-Ur_{2n}\ ||^2}{1+2\ ||r_{2n}-s_n||^2 + 2||Vs_{2n}-r_{2n}||^2} + b\ \frac{[1+2||Ur_{2n}-Vs_{2n}||^2\ (2\ ||r_{2n}-Vs_{2n}\ ||^2 + 2||Vs_{2n}-Ur_{2n}||^2)}{1+2\ ||r_{2n}-s_n||^2 + 2||Vs_{2n}-r_{2n}||^2} + c\ [2||Ur_{2n}-Vs_{2n}||^2 + 2||Vs_{2n}-r_{2n}||^2]$$

$$\leq 2a[||Ur_{2n}-Vs_{2n}||^2 + ||r_{2n}-Vs_{2n}\ ||^2]$$

$$+ 2b[||r_{2n} - Vs_{2n}||^2 + ||Vs_{2n} - Ur_{2n}||^2]$$

$$+ 2c [||Ur_{2n} - Vs_{2n}||^2 + ||Vs_{2n} - r_{2n}||^2]$$

$$\leq \frac{(a+b+c)}{1-2(a+b+c)} || r_{2n} - Vs_{2n}||^2$$

Taking the lim as $n \to \infty$, we get $||Ur_{2n} - Vs_{2n}||^2 \to 0$. It follows that

$$||r_{2n} - Ur_{2n}||^2 \leq ||r_{2n} - Vs_{2n}||^2$$

$$+ 2||Vs_{2n} - Ur_{2n}||^2 \to 0$$

and,

$$||u - Ur_{2n}||^2 \leq 2||u - r_{2n}||^2$$

$$+ 2||x_{2n} - Vy_{2n}||^2 \to 0 \; as \; \to \infty.$$

If r_{2n} and u satisfy (2.1) we have

$$||Ur_{2n} - Vu||^2 \leq a \frac{||u-Vu||^2[1+||r_{2n}-Ur_{2n}||^2]}{1+||r_{2n}-u||^2}$$

$$+ b \frac{[1+||u-Vu||^2]||r_{2n}-Ur_{2n}||^2}{1+||r_{2n}-u||^2}$$

$$+ c ||r_{2n} - u||^2$$

$$\leq a \frac{||u-r_{2n}+r_{2n}-Vu||^2[1+||r_{2n}-Ur_{2n}||^2]}{1+||r_{2n}-u||^2}$$

$$+ b \frac{[1+||u-r_{2n}+r_{2n}-Vu||^2]||r_{2n}-Ur_{2n}||^2}{1+||r_{2n}-u||^2}$$

$$+ c ||r_{2n} - u||^2$$

$$\leq a \frac{\left[2||u-r_{2n}||^2+2||r_{2n}-Vu||^2\right][1+||r_{2n}-Vu||^2]}{1+||r_{2n}-u||^2}$$

$$+ b \frac{1+\left[2||u-r_{2n}||^2+2||r_{2n}-Vu||^2\right]||r_{2n}-Ur_{2n}||^2}{1+||r_{2n}-u||^2}$$

$$+ c ||r_{2n} - u||^2$$

$$\leq a \frac{\left[2||u-r_{2n}||^2+4||r_{2n}-Ur_{2n}||^2\right]4+||Ur_{2n}-Vu||^2][1+||r_{2n}-Vu||^2}{1+||r_{2n}-u||^2}$$

$$+ b \frac{1+\left[2||u-r_{2n}||^2+4||r_{2n}-Ur_{2n}||^2\right]+4||Ur_{2n}-Vu||^2[1+||r_{2n}-Ur_{2n}||^2]}{1+||r_{2n}-u||^2}$$

$$+ c ||r_{2n} - u||^2$$

Taking the lim as $n \to \infty$, we obtain

$$||Ur_{2n} - Vu||^2 \leq (a+b) ||Ur_{2n} - Vu||^2$$

$$(1-a-b) ||Ur_{2n} - Vu||^2 \leq 0.$$

that is, $||Ur_{2n} - Vu||^2 \to 0$

Finally $||u-Vu||^2 = || u - Ur_{2n} + Ur_{2n} - Vu||^2$

$\leq 2|| u - Ur_{2n} ||^2 + || Ur_{2n} - Vu||^2 \to 0$ as $n \to \infty$,

showing that u= Vu.

Similarly, we can prove that u= Uu.

Thus, u is a common fired point of U and V.

THEOREM 2.2.

Let B be a Hilbert space and C be a closed subset of B. Let U and V be two sets of mappings satisfying for some positive integers with

$$||U'r - V's||^2 \leq \mu \frac{||s-V's||^2[1+||r-U'r||^2]}{1+||r-s||^2} + \omega ||r-s||^2 \quad \text{....... (3.1)}$$

where $0 \leq 3\mu + 3\omega \leq 1$.. If there exists a point r_0 such that the I- scheme for U and V defined by (1.2) (1.3), converges to a point u, then u is a common fired point of U and V.

PROOF:

It follows from (1.2)

$r_{2n+1} - r_{2n} = \mu_{2n}(Vs_{2n} - r_{2n})$.

Since $r_{2n} \to u$. $|| r_{2n+1} - r_{2n} || \to 0$. Since $\{ \mu_{2n} \}$is bounded away from zero , $|| Vs_{2n} - r_{2n}|| \to 0$.

It follows that $||u - Vs_{2n}|| \to 0$. Since U and V satisfs (3.1) we have,

$$||U'r_{2n} - V's_{2n}||^2 \leq \alpha \frac{||s_{2n}-V's_{2n}||^2[1+||r_n-U'r_{2n}||^2]}{1+||r_{2n}-s_{2n}||^2}$$

$$+ \beta ||r_{2n} - s_{2n}||^2 \quad (3.2)$$

Now

$||s_{2n} - r_{2n}||^2 = || \omega_{2n} U'r_{2n} + (1- \omega_{2n}) r_{2n} - r_{2n}||^2$

$\leq 2||U'r_{2n} - V's_{2n}||^2 + 2 ||V's_{2n} - r_{2n}||^2$ (3.3)

$||s_{2n} - V's_{2n}||^2 \leq 2||U'r_{2n} - V's_{2n}||^2 + 2 || r_{2n} - V's_{2n} ||^2$ (3.4)

From (3.2) (3.3) (3.4) can be written as

$||U'r_{2n} - V's_{2n}||^2 \leq$

$$\alpha \frac{\left[2\left||U'r_{2n} - V'^{s_{2n}}|\right|^2 + 2\left||r_n - V'^{s_{2n}}|\right|^2\right]\left[1 + 2\left||r_n - V'^{s_{2n}}|\right|^2 + 2\left|V'^{s_{2n}} - U'^{r_{2n}}|\right|^2\right]}{\left[1 + 2\left||U'^{r_{2n}} - V'^{s_{2n}}|\right|^2 + 2\left||V'^{s_{2n} - r2n}|\right|^2\right]}$$

$+ \beta [2 \left||U'r_{2n} - V'^{s_{2n}}|\right|^2 + 2 \left||V'^{s_{2n} - r_{2n}}|\right|^2]$

$\leq \mu [2\left||U'r_{2n} - V'^{s_{2n}}|\right|^2 + 2 || r_{2n} - V's_{2n} ||^2] + \omega [2 \left||U'r_{2n} - V'^{s_{2n}}|\right|^2 + 2 \left||V'^{s_{2n} - r_{2n}}|\right|^2]$

$||U'r_{2n} - V's_{2n}||^2 - 2(\mu + \omega) \left||U'r_{2n} - V'^{s_{2n}}|\right|^2$

$\leq 2 + (\mu + \omega) ||r_{2n} - V's_{2n}||^2$ 1-2 $(\mu + \omega) ||U'r_{2n} - V's_{2n}||^2$

$\leq 2 + (\mu + \omega) ||r_{2n} - V's_{2n}||^2$

$||U'r_{2n} - V's_{2n}||^2 \leq \frac{2(\mu + \omega)}{1-2(\mu + \omega)} ||r_{2n} - V's_{2n}||^2$

Taking lim as $n \to \infty$, we get $||U'r_{2n} - V's_{2n}||^2 \to 0$

It follows that,

$||r_{2n} - U'r_{2n}||^2 \leq 2 ||r_{2n} - V's_{2n}||^2 + 2 ||V'r_{2n} - U's_{2n}||^2 \to 0$

and,

$||u - U'r_{2n}||^2 \leq 2 ||u - r_{2n}||^2 + 2 ||r_{2n} - V's_{2n}||^2 \to 0$ as $n \to \infty$,

If r_{2n} and u satisfy (3.1) we have

$||U'r_{2n} - V'u||^2 \to 0$

Finally $|| u - V'u||^2 = ||u - U'r_{2n} + U'r_{2n} - V'u||^2$

$\leq 2 ||u - U'r_{2n}||^2 + 2 ||U'r_{2n} - V'u||^2 \to 0$ as $n \to \infty$,

showing that u = V'u.

Similarly, we can prove that u = U'u.

Thus u is a common fired point of U and V.

3. CONCLUSION

In this work ,we have proved back to back two theorems for a unique common fixed point theorem in Hilbert space by functional type inequality . It may be enhanced for various inequality , weak and strong convergence , taking three and more mappings etc.

REFERENCES

[1] Lj.B.Ciric, Generalized contraction and fixed point theorem, Publ .Inst. Math. Vol. 12(26),19-26,1971.

`[2] B.K.Das and S.Gupta, An extension of Banach contraction principal through rational expression , Indian Journal of Pure and Applied Math. Vol.6, 1455-1458,1975.

[3] M. Imdad and J.Ali , A note on a fixed point theorem in Hilbert spaces, Thai Journal of Mathematics Vol. 3 ,No. 2 : 219–221, 2005.

[4] S.K.Jain , and S. A., Sayyed , Nonlinear contraction satisfying square inequality and fixed point theory, International Journal of Research and Analytical Reviews , Vol. 6, Issue 2,pp.923-929, 2019.

[5] S.P.Mane and S.A.Sayyed, Study on contractive maps with different type of fixed point, The Journal of Oriental Research Madras,Vol.MMXXII-XCIII-II

[6] S.A.Naimpally and K.L. Singh, Extensions of some fixed point theorems of Rhoades, J.Math.Anal.Vol.96, 437-446,1983.

[7] S. K. Nigam, N. P. S. Bawa and P. K. Shrivastava, Fixed point theorems in

Hilbert space, Varhahmihir J. Math. Sci.,Vol. 3(1), 113-118,2003.

[8]S. Park, Fixed points and periodic points of contractive pairs of maps, Proc.College Nat. Sci., Seoul Nat. Univ., Vol.5(1) 9-22, 1980.

[9] M. Patel and S. Sharma, Fixed point theorem in hilbert space using weak and strong convergence, Global Journal of Pure and Applied Mathematics, Vol.13, No. 9 pp. 5183-5193, 2017.

[10] N.Rao , K.Kalyani and K.V. Acharyulu, A unique fixed point theorem in Hilbert space. Acta Cienc Indic Vol.41pp.39-46. 2015.

[11] N. S.Rao and K. Kalyani , Some results on common fixed point theorems in Hilbert space, Asian Journal of Mathematical Sciences, Vol. 4,Issue 1 pp.66-74,2020

[12] B. E. Rhoades, A comparison of various definitions of contractive mappings,Trans. Amer. Math. Soc., Vol.226,pp.257-290, 1977.

[13] B. E. Rhoades, Contraction definitions revisited, Contemporary Math., Vol.21, pp189-205,1983.

[14] F. Sayyed and V.H.Badshah, Generalized contraction and common fixed point theorem in Hilbert space, J.Indian.Acad.Math.,Vol.23,No.2,pp-267-275, 2001.

[15] S.A.sayyed and V.H.Badshah, Generalization of common fixed point theorem of Naimpally and Singh, Ultra Science, Vol.17(3), pp.461-464, 2005.

[16] S.A.sayyed and V.H.Badshah, Common Fixed Point Iteration Process In Hilbert Space , Vikram Mathematical Journal ,Vol.25,pp. 160-165,2005.

[17]S.A.sayyed and V.H.Badshah, Extensions Of Some Common Fixed Point Theorems In Hilbert Space". Jnanabha, Vol.35, pp.,2005.

[18] S.A Sayyed,. K. Devghare, and V.H Badshah, , ,A Note On Fixed Point For Selfmaps, *Acta Ciencia Indica, Vol. XXXII M, No. 4, pp.1595-1596,2006.*

[19] S.A., Sayyed, ,L Vyas., F. Sayyed , And V.H Badshah,., Some Results On Fixed Point Theorems For Selfmaps Ultra Engineer , Vol.1 (2),pp143-145,2012.

[20] T.Veerapandi and S.A. Kumar ,Common fixed point theorems of a sequence of mappings on Hilbert space. Bull Calcutta Math Soc Vol.91,pp.299-308, 1999.

[21]H. Yadav , S.A.sayyed and V.H.Badshah ,A Note On Common Fixed Point Theorem In Hilbert Space ,Material Science Research India Vol .7 No.2,pp.515-518, 2010.